Introduction to Effectus Theory

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Outline

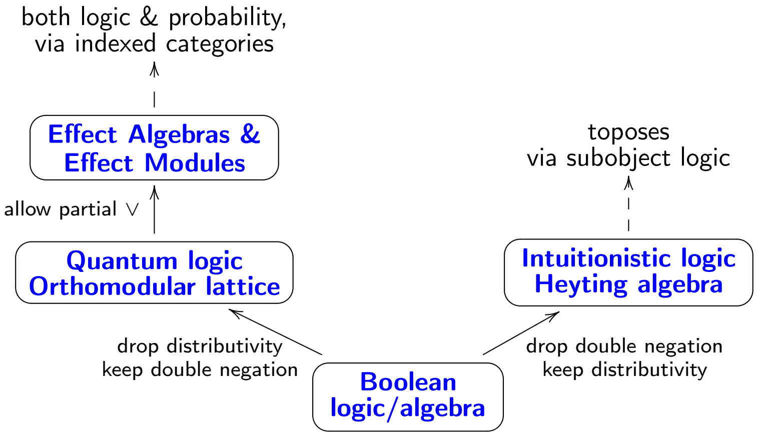
Where we are, so far

About this talk

- Overview of quantum logic research at Nijmegen
- Performed within context of ERC Advanced Grant Quantum Logic, Computation, and Security
    - Running period: 1 May 2013 – 1 May 2018
- Focus on categorical axiomatisation of the quantum world
    - esp. differences/similarties with probabilistic and classical computing
- Key notion is effectus, a special kind of category (see later)

Group picture

From Boolean to intuitionistic & quantum logic



both logic & probability,
via indexed categories

**Effect Algebras &
Effect Modules**

allow partial ∨

**Quantum logic
Orthomodular lattice**

toposes
via subobject logic

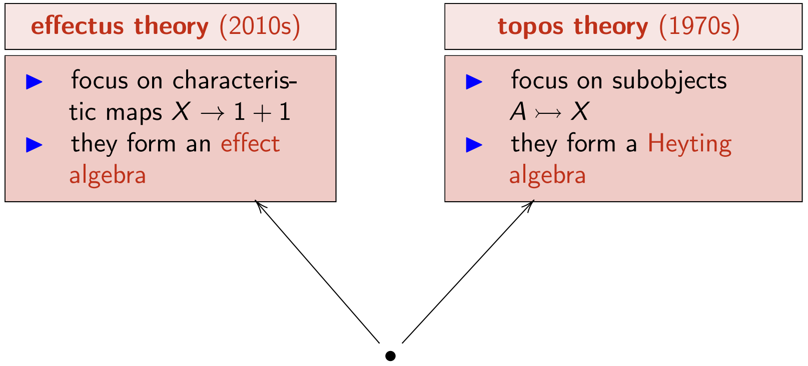
**Intuitionistic logic
Heyting algebra**

drop distributivity
keep double negation

**Boolean
logic/algebra**

drop double negation
keep distributivity

Aha-moments in categorical logic



effectus theory (2010s)

- focus on characteristic maps $X \to 1+1$
- they form an effect algebra

topos theory (1970s)

- focus on subobjects $A \rightarrowtail X$
- they form a Heyting algebra

Example (without knowing yet what an effectus is)

The opposite $\mathbf{Rng}^{\mathrm{op}}$ of the category of rings (with unit) is an effectus, with:

$$\begin{array}{cl} R \xrightarrow{\text{predicate}} 1+1 & \text{in } \mathbf{Rng}^{\mathrm{op}} \\ \hline\hline \mathbb{Z}\times\mathbb{Z} \longrightarrow R & \text{in } \mathbf{Rng} \\ \hline\hline \text{idempotent } e \in R\text{, so } e^2 = e & \end{array}$$

Hence the predicates on $R \in \mathbf{Rng}^{\mathrm{op}}$ are its idempotents

- These idempotents $e \in R$ form an effect algebra, with:

  truth 1   falsum 0   orthocomplement $e^{\perp} = 1 - e$

  Additionally there is a partial sum $e \ovee d = e + d$ if $ed = 0 = de$.
- If R is commutative, then the idempotents form a Boolean algebra!
  (this case is well-known/studied, eg. in sheaf theory for commutative rings)

Origin of 'effectus'

New Directions paper

- B. Jacobs, *New Directions in Categorical Logic, for Classical, Probabilistic and Quantum Logic*, LMCS 11(3), 2015
- Introduces four successive assumptions (and elaborates them)

Intro paper

- Cho, Jacobs, Westerbaan, Westerbaan, *Introduction to Effectus Theory*, 2015, `arxiv.org/abs/1512.05813`, 150p.

Several other papers by ERC team members, eg.

- Kenta Cho, on equivalence between 'total' and 'partial' description
- Robin Adams, on "effect" logic & type theory
- Bas & Bram Westerbaan, on von Neumann algebra model

Where we are, so far

Effect algebras, definition

Effect algebras axiomatise the unit interval $[0, 1]$ with its (partial!) addition $+$ and "negation" $x^\perp = 1 - x$.

Definition

A Partial Commutative Monoid (PCM) consists of a set M with zero $0 \in M$ and partial operation $\ovee \colon M \times M \to M$, which is suitably commutative and associative.

One writes $x \perp y$ if $x \ovee y$ is defined.

Definition

An effect algebra is a PCM in which each element x has a unique 'orthosuplement' $x^\perp$ with $x \ovee x^\perp = 1$ $(= 0^\perp)$
Additionally, $x \perp 1 \Rightarrow x = 0$ must hold.

Effect algebras, observations

- There is then a partial order, via $x \leq y$ iff $y = x \ovee z$, for some z
- Each Boolean algebra is an effect algebra, with:

$$x \perp y \text{ iff } x \wedge y = 0, \qquad \text{and then} \qquad x \ovee y = x \vee y$$

- In fact, each orthomodular lattice is an effect algebra (in the same way)
- Frequently occurring form: unit intervals:

$$[0, 1]_G = \{x \in G \mid 0 \leq x \leq 1\}$$

  in an ordered Abelian group with order unit $1 \in G$.
  - $x^\perp = 1 - x$
  - $x \perp y$ iff $x + y \leq 1$, and in that case $x \ovee y = x + y$.

Homomorphisms of effect algebras

Definition

A homomorphism of effect algebras $f \colon X \to Y$ satisfies:

- $f(1) = 1$
- if $x \perp x'$ then both $f(x) \perp f(x')$ and $f(x \ovee x') = f(x) \ovee f(x')$.

This yields a category **EA** of effect algebras.

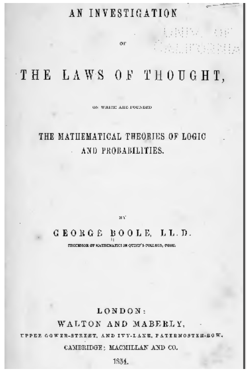
Example:

- A probability measure yields a map $\Sigma_X \to [0,1]$ in **EA**
- Recall the indicator (characteristic) function $\mathbf{1}_U \colon X \to [0,1]$, for a subset $U \subseteq X$.
  - It gives a map of effect algebras:

$$\mathcal{P}(X) \xrightarrow{\mathbf{1}_{(-)}} [0,1]^X$$

Naturality of partial sums/disjunctions in logic

George Boole in 1854 thought of disjunction as a partial operation



AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

BY

GEORGE BOOLE, LL.D.
PROFESSOR OF MATHEMATICS IN QUEEN'S COLLEGE, CORK.

LONDON:
WALTON AND MABERLY,
UPPER GOWER-STREET, AND IVY-LANE, PATERNOSTER-ROW.
CAMBRIDGE: MACMILLAN AND CO.
1854.

"Now those laws have been determined from the study of instances, in all of which it has been a necessary condition, that the classes or things added together in thought should be mutually exclusive. The expression $x + y$ seems indeed uninterpretable, unless it be assumed that the things represented by x and the things represented by y are entirely separate; that they embrace no individuals in common." (p.66)

Effect modules

Effect modules are effect algebras with a scalar multiplication, with scalars not from $\mathbb{R}$ or $\mathbb{C}$, but from $[0,1]$.

(Or more generally from an "effect monoid", ie. effect algebra with multiplication)

Definition

An effect module M is a effect algebra with an action $[0,1] \times M \to M$ that is a "bihomomorphism"

A map of effect modules is a map of effect algebras that commutes with scalar multiplication.

We get a category **EMod** $\hookrightarrow$ **EA**.

Effect modules, main examples

Probabilistic examples

- Fuzzy predicates $[0,1]^X$ on a set X, with scalar multiplication

$$r \cdot p \overset{\text{def}}{=} x \mapsto r \cdot p(x)$$

- Measurable predicates $\mathrm{Hom}(X, [0,1])$, for a measurable space X, with the same scalar multiplication
- Continuous predicates $\mathrm{Hom}(X, [0,1])$, for a topological space X

Quantum examples

- Effects $\mathcal{E}(H)$ on a Hilbert space: operators $A \colon H \to H$ satisfying $0 \leq A \leq I$, with scalar multiplication $(r, A) \mapsto rA$.
- Effects in a C^*/W^*-algebra A: positive elements below the unit:

$$[0,1]_A = \{a \in A \mid 0 \leq a \leq 1\}.$$

This one covers the previous illustrations.

Basic adjunction, between effects and states

Theorem By "homming into $[0,1]$" one gets an adjunction:

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[r, bend left, "\mathrm{Hom}(-,[0,1])"] \arrow[r, phantom, "\top"] & \mathbf{Conv} \arrow[l, bend left, "\mathrm{Hom}(-,[0,1])"]
\end{tikzcd}$$

This adjunction restricts to an equivalence of categories between:

- Banach effect modules, which have a complete norm
  (or equivalently, complete order unit spaces)
- convex compact Hausdorff spaces

This is called **Kadison duality**

Where we are, so far

Background

A crash course on effect algebras and effect modules

Effectuses

Basic results in effectus theory

Tool support for effectus probability

Conclusions

Effectus

An **effectus** is a category with finite coproducts $(0,+)$ and 1 such that

- these diagrams are pullbacks:

$$\begin{tikzcd}
A+X \arrow[r, "\mathrm{id}+g"] \arrow[d, "f+\mathrm{id}"'] & A+Y \arrow[d, "f+\mathrm{id}"] \\
B+X \arrow[r, "\mathrm{id}+g"'] & B+Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{id}"] \arrow[d, "\kappa_1"'] & A \arrow[d, "\kappa_1"] \\
A+X \arrow[r, "\mathrm{id}+f"'] & A+Y
\end{tikzcd}$$

- these arrows are jointly monic:

$$\begin{tikzcd}
X+X+X \arrow[r, shift left, "|\!|\!/ = {[\kappa_1,\kappa_2,\kappa_2]}"] \arrow[r, shift right, "\backslash\!\!/ = {[\kappa_2,\kappa_1,\kappa_2]}"'] & X+X
\end{tikzcd}$$

Perspective:

$$\left(\begin{array}{c}\text{disjoint and universal}\\ \text{coproducts}\end{array}\right) \Longrightarrow \left(\text{effectus}\right) \Longrightarrow \left(\begin{array}{c}\text{disjoint}\\ \text{coproducts}\end{array}\right)$$

Internal logic

effectus	**meaning**
objects X	types
arrows $X \overset{f}{\to} Y$	programs
1 (final object)	singleton/unit type
$1 \xrightarrow{\omega} X$	state
$X \xrightarrow{p} 1+1$	predicate
$\begin{tikzcd} 1 \arrow[r, "\omega"] \arrow[rr, bend right, "\omega\models p"'] & X \arrow[r, "p"] & 1+1 \end{tikzcd}$	validity $\omega \models p$
$1 \longrightarrow 1+1$	scalar
$f_*(\omega) = f \circ \omega$	state transformation
$f^*(q) = q \circ f$	predicate transformation

$$\boxed{\begin{array}{c} f_*(\omega) \models q \\ = \\ \omega \models f^*(q) \end{array}}$$

Discrete probability example

- **Claim**: $\mathcal{K}\ell(\mathcal{D})$ is an effectus!
- **Question**: What are the predicates and states?

- **Predicates** are maps $p\colon X \to 1+1 = 2$ in $\mathcal{K}\ell(\mathcal{D})$
  - hence they are functions $p\colon X \to \mathcal{D}(2) \cong [0,1]$
  - predicates on X in $\mathcal{K}\ell(\mathcal{D})$ are thus fuzzy: elements of $[0,1]^X$
- **States** are maps $\omega\colon 1 \to X$ in $\mathcal{K}\ell(\mathcal{D})$
  - hence functions $1 \to \mathcal{D}(X)$, or elements of $\mathcal{D}(X)$
  - and so discrete probability distributions on X
- **Validity** $\omega \models p$ is Kleisli composition $p \circ \omega\colon 1 \to 1+1$
  - the outcome is a probability in $\mathcal{D}(2) \cong [0,1]$
  - it is given by the expected value $\sum_x \omega(x) \cdot p(x)$

Examples of states and predicates in an effectus

	State	**Predicate**	**Validitv**	**Scalars**
	$1 \overset{\omega}{\to} X$	$X \overset{p}{\to} 1+1$	$\omega \models p$	$1 \to 1+1$
classical **Sets**	element $\omega \in X$	subset $p \subseteq X$	$\omega \in p$	$\{0,1\}$
probabilistic $\mathcal{K}\ell(\mathcal{D})$	discrete distribution $\omega \equiv \sum_i s_i \,\lvert x_i\rangle$	fuzzy predicates $X \overset{p}{\to} [0,1]$	$\sum_i s_i p(x_i)$	$[0,1]$
probabilistic $\mathcal{K}\ell(\mathcal{G})$	probability measure $\Sigma_X \overset{\phi}{\to} [0,1]$	measurable predicates $X \overset{p}{\to} [0,1]$	$\int p d\phi$	$[0,1]$
quantum $\mathbf{vNA}^{\mathrm{op}}$	normal state $\omega\colon X \to \mathbb{C}$	effect $0 \leq p \leq 1$ in X	$\omega(p)$	$[0,1]$

Effect structure on predicates $X \to 1+1$

- We get some logical structure for free:

$$1 = \big(X \xrightarrow{\kappa_1 \circ !} 1+1\big) \quad 0 = \big(X \xrightarrow{\kappa_2 \circ !} 1+1\big) \quad p^\perp = \big(X \xrightarrow{p} 1+1 \xrightarrow[\cong]{[\kappa_2,\kappa_1]} 1+1\big)$$

  Then $p^{\perp\perp} = p$, $0^\perp = 1$, $1^\perp = 0$.

- Define $p \perp q$, for $p, q\colon X \to 1+1$ if there is a bound b in:

$$\begin{tikzcd}
 & X \arrow[dl, "p"'] \arrow[d, dashed, "b"] \arrow[dr, "q"] & \\
1+1 & 1+1+1 \arrow[l, "\rhd_1"'] \arrow[r, "\rhd_2"] & 1+1
\end{tikzcd}$$

  In that case put $p \ovee q = (\nabla + \mathrm{id}) \circ b\colon X \to 1+1$.

- Predicates $1 \to 1+1$ on 1 will be called scalars
  - they carry a monoid structure $p \cdot q = [p, \kappa_2] \circ q$
  - it is commutative in presence of distributive tensors

The structure of predicates and states

Theorem

Let $\mathbf{B}$ be an effectus. Then:

(1) The predicates $X \to 1+1$ form an effect module

(2) The states $1 \to X$ form a convex set

Predicate transformers f^* and state transformers f_* preserve this structure. We get a state-and-effect triangle:

$$\begin{tikzcd}
\mathbf{EMod}^{\mathrm{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathbf{Conv} \arrow[ll, bend left] \\
 & \mathbf{B} \arrow[ul, "Pred"] \arrow[ur, "Stat"'] &
\end{tikzcd}$$

General picture: “state-and-effect triangles”

$$\begin{tikzcd}[column sep=small]
\textbf{Heisenberg} & & \textbf{Schrödinger} \\
\left(\begin{array}{c}\text{predicate}\\ \text{transformers}\end{array}\right)^{\text{op}} \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \left(\begin{array}{c}\text{state}\\ \text{transformers}\end{array}\right) \arrow[ll, bend left] \\
& \left(\text{computations}\right) \arrow[ul, "\text{Pred}"] \arrow[ur, "\text{Stat}"'] &
\end{tikzcd}$$

- The traditional distinction in program semantics between predicate transformers and state transformers also exists in the quantum world
- It corresponds to the different approaches of Heisenberg (matrix mechanics) and Schrödinger (wave equation, for pure state changes)

Where we are, so far

Background

A crash course on effect algebras and effect modules

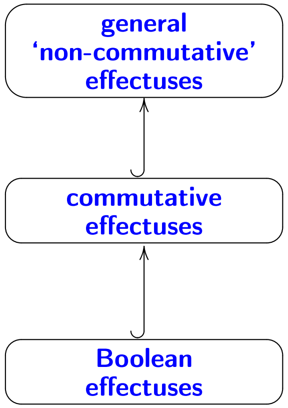
Effectuses

Basic results in effectus theory
  Effectuses for probability and classical computation
  Assert maps for sequential conjunction and conditioning
  Quotients and comprehension

Tool support for effectus probability

Conclusions

Overview: subclasses of effectuses



general ‘non-commutative’ effectuses

von Neumann algebras

commutative effectuses

commutative von Neumann algebras, $\mathcal{K}\ell(\mathcal{D}), \mathcal{K}\ell(\mathcal{G})$

Boolean effectuses

Sets, extensive categories

Defining these subclasses, I

Definition

A map $f\colon X \to X+1$ is called side-effect free if $f \leq \mathrm{id}$, where:
- $\mathrm{id} = \kappa_1\colon X \to X+1$ is the Kleisi/partial identity map
- $\leq$ is an ‘obvious’ order on partial maps, defined as for predicates

Note: we can always turn a partial map into a predicate:

$$\left(X \xrightarrow{f} X+1\right) \longmapsto \left(X \xrightarrow{f} X+1 \xrightarrow{!+\mathrm{id}} 1+1\right)$$

- Often, one can also go the other way around: from predicates to partial endomaps
- This inverse is called assert, written as asrt_p for predicate p
- Sometimes this assert map is even side-effect free.

Defining these subclasses, II

Definition

The effectus **B** is called commutative if

- there are side-effect free inverses asrt_p for "partial-map-to-predicate"
- these assert maps commute: $\mathrm{asrt}_p \circ \mathrm{asrt}_q = \mathrm{asrt}_q \circ \mathrm{asrt}_p$

An effectus is Boolean if it is commutative and assert maps are idempotent: $\mathrm{asrt}_p \circ \mathrm{asrt}_p = \mathrm{asrt}_p$.

Main results

Theorem

- *In a commutative effectus, $Pred(X)$ is a commutative effect monoid*
- *In a Boolean effectus, $Pred(X)$ is a Boolean algebra, functorially:*

$$\mathbf{B} \xrightarrow{Pred} \mathbf{BA}^{op}$$

Theorem

Boolean effectuses 'with comprehension' are the same as extensive categories

An extensive category has 'well-behaved' coproducts: they are disjoint and universal.

Assert maps for sequential conjunction ('andthen')

- For two predicates $p, q \colon X \to 1+1$ define sequential conjunction:

$$p \mathbin{\&} q \;:=\; \Big(X \xrightarrow{\mathrm{asrt}_p} X+1 \xrightarrow{[q,\kappa_2]} 1+1\Big)$$

- This $p \mathbin{\&} q$ incorporates the side-effect of p, via its assert map
  - indeed, & is non-commutative in general, in the quantum case
  - but it is commutative in commutative effectuses (probabilistic case)
- More concretely,
  - for $p, q \in [0,1]^X$ we have $(p \mathbin{\&} q)(x) = p(x) \cdot q(x)$
  - for $p, q \in \mathcal{B}(\mathscr{H})$, we use $p \mathbin{\&} q = \sqrt{p}q\sqrt{p}$

Assert maps for conditioning of states

- Assert maps are also useful for conditioning of states
  - conditioning is also called (Bayesian) state update/revision
  - a uniform description can be given in an effectus
  - it requires normalisation, of partial states to proper states
- Let $\omega \colon 1 \to X$ be state, and $p \colon X \to 1+1$ a predicate
  - we get a partial state by composition:

$$1 \xrightarrow{\omega} X \xrightarrow{\mathrm{asrt}_p} X+1$$

  - write $\omega|_p \colon 1 \to X$ for its normalisation; it exits if $\omega \models p \neq 0$
  - Read $\omega|_p$ as: ω, given p
- Once prove the conditional probability rule:

$$\omega|_p \models q \quad = \quad \frac{\omega \models p \mathbin{\&} q}{\omega \models p}$$

About quotients and comprehension

- Familiar picture in categorical logic:

  truth $\dashv$ comprehension

- Quotients X/R defined for relations $R \subseteq X \times X$ give:

  quotients $\dashv$ equality

- In linear algebra quotients A/S are typically defined for subspaces $S \subseteq A$. Then:

  quotients $\dashv$ falsity

Recall that truth and falsity predicates form right and left adjoints to a fibration (functor), giving a quotient-comprehension chain:

$$\boxed{\text{quotients} \dashv \text{falsity} \dashv \text{fibration} \dashv \text{truth} \dashv \text{comprehension}}$$

Example chains

- For vector spaces:

$$\begin{tikzcd}
LSub(\mathbf{Vect}) \arrow[d, bend right=80, "{S\subseteq V \atop \mapsto V/S}"'] \arrow[d] \arrow[d, bend left=80, "{S\subseteq V \atop \mapsto S}"] \\
\mathbf{Vect} \arrow[u, bend left=40, "0"] \arrow[u, bend right=40, "1"'] \arrow[u, phantom, shift left=6ex, "\dashv"] \arrow[u, phantom, shift left=2ex, "\dashv"] \arrow[u, phantom, shift right=2ex, "\dashv"] \arrow[u, phantom, shift right=6ex, "\dashv"]
\end{tikzcd}$$

- For Hilbert spaces:

$$\begin{tikzcd}
CLSub(\mathbf{Hilb}) \arrow[d, bend right=80, "{S\subseteq V \atop \mapsto S^{\perp}}"'] \arrow[d] \arrow[d, bend left=80, "{S\subseteq V \atop \mapsto S}"] \\
\mathbf{Hilb} \arrow[u, bend left=40, "0"] \arrow[u, bend right=40, "1"'] \arrow[u, phantom, shift left=6ex, "\dashv"] \arrow[u, phantom, shift left=2ex, "\dashv"] \arrow[u, phantom, shift right=2ex, "\dashv"] \arrow[u, phantom, shift right=6ex, "\dashv"]
\end{tikzcd}$$

- Each Abelian category $\mathbf{A}$ has:

$$\begin{tikzcd}
Sub(\mathbf{A}) \arrow[d, bend right=80] \arrow[d] \arrow[d, bend left=80] \\
\mathbf{A} \arrow[u, bend left=40, "0"] \arrow[u, bend right=40, "1"'] \arrow[u, phantom, shift left=6ex, "\dashv"] \arrow[u, phantom, shift left=2ex, "\dashv"] \arrow[u, phantom, shift right=2ex, "\dashv"] \arrow[u, phantom, shift right=6ex, "\dashv"]
\end{tikzcd}$$

Effectuses with quotient comprehension chains

For an effectus $\mathbf{B}$ write:

- $PMap(\mathbf{B})$ for the category of partial maps $X \to Y+1$ in $\mathbf{B}$
- $PPred(\mathbf{B})$ for the category with predicates $p\colon X \to 1+1$ as objects.
  - maps $(X \overset{p}{\to} 1+1) \xrightarrow{f} (Y \overset{q}{\to} 1+1)$ are $f\colon X \to Y+1$ with:

$$p \quad \leq \quad (q^{\perp} \circ f)^{\perp}$$

Definition

An effectus has quotient and comprehension if there are outer adjoints:

$$\begin{tikzcd}
PPred(\mathbf{B}) \arrow[d, bend right=80] \arrow[d] \arrow[d, bend left=80] \\
PMap(\mathbf{B}) \arrow[u, bend left=40, "0"] \arrow[u, bend right=40, "1"'] \arrow[u, phantom, shift left=6ex, "\dashv"] \arrow[u, phantom, shift left=2ex, "\dashv"] \arrow[u, phantom, shift right=2ex, "\dashv"] \arrow[u, phantom, shift right=6ex, "\dashv"]
\end{tikzcd}$$

Such chains exist in all leading examples: non-trivial for v. Neumann algebras

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Quotient-comprehension chains and measurement

- It turns out that there are close connections between:
  - quotient-comprehension chains in an effectus
  - measurement, via "side-effectful" assert maps
- Canonical form in von Neumann algebras: $\mathrm{asrt}_p(x) = \sqrt{p}\cdot x \cdot \sqrt{p}$
- In all our examples we can factor assert (as partial map):

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{asrt}_p"] \arrow[dr, "\xi_{p^{\perp}}"'] & & X \\
& X/p^{\perp} \cong \{X \mid \lceil p \rceil\} \arrow[ur, "\pi_{\lceil p \rceil}"'] &
\end{tikzcd}$$

This is formalised in a telos:

- an effectus with a square root axiom
- it axiomatises von Neumann algebras — and quantum theory
- details are still forthcoming

Where we are, so far

EfProb tool support, see `efprob.cs.ru.nl`

- EfProb is abbreviation of *Effectus Probability*
  - developed jointly with Kenta Cho
- It is an embedded language of Python, for probabilistic calculations
  - it yields channel-based probability theory
  - abstractly: a channel is a map in an effectus
  - concretely: conditional probability, stochastic matrix, Markov kernel, . . .
- EfProb uses: states, predicates, random variables, validity, conditioning, state- and predicate-transformation, disintegration . . .
  - uniform terminology & notation for discrete/continuous/quantum
  - think: $\mathcal{K}\ell(\mathcal{D})$ / $\mathcal{K}\ell(\mathcal{G})$ / $\mathbf{vNA}^{\mathrm{op}}$
- Extensive manual is available, with many, many examples
  - Bayesian networks, hidden Markov models, quantum protocols, . . .

Example: fish in a pond

Capture-recapture challenge

Imagine we wish to estimate the number of fish in a pond.

(1) we catch 20 fish, mark them, and throw them all back

(2) we wait a bit, catch 25, and find 5 are marked.

How many fish are in the pond?

Assumptions for the mathematical model

- the range of fish is $[25, 300]$, as *continuous* interval
- the prior distribution is uniform
- in (2), each observed fish is thrown back before another is caught
- thus we can use a binomial with $N = 25$, and probability $p = \frac{20}{x}$, where $x \in [20, 300]$ is the number of fish

---

Fish example in EfProb

Define domains (sample spaces) and priors:

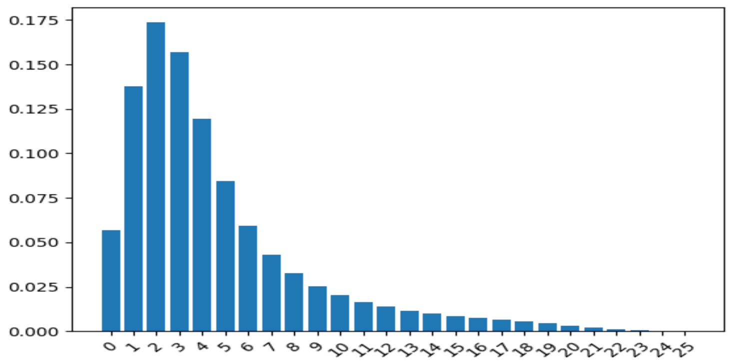
```
>>> fish_dom = R(25,300)
>>> catch_dom = range(0,26)
>>> prior = uniform_state(fish_dom)
```

Next, a channel $[25, 300] \to \mathcal{D}(\{0, \ldots, 25\})$

```
>>> c = chan_fromklmap(lambda x: binomial(25, 20/x),
...                    fish_dom, catch_dom)
>>> catch = c >> prior  # forward state transformation
>>> catch.plot()        # draw picture
```

State transformation >> gives (Bayesian) prediction

Predict probability of catching n marked fish

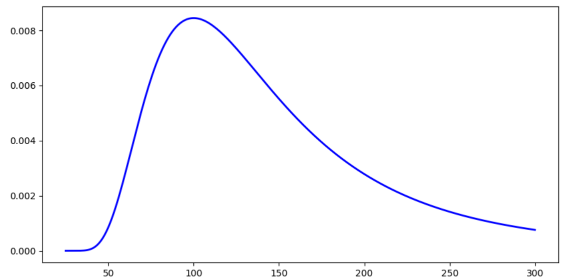


A discrete probability distribution on $\{0, \ldots, 25\}$, assuming the prior uniform distribution on $[25, 300]$.

Catch 25 fish, find 5 marked: reason backwards

Define 'observe 5' predicate, then transform this predicate & condition:

```
>>> obs_5 = point_pred(5,catch_dom)
>>> post_5 = prior / (c << obs_5)
>>> post_5.plot()
```

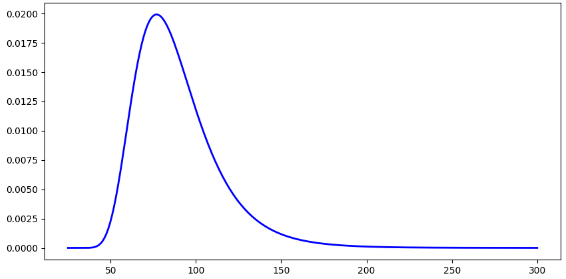


The expected number of fish is 139

Catch another 25 fish, now find 8 marked

Update the earlier posterior state `post_5` once again:

```
>>> obs_8 = point_pred(8,catch_dom)
>>> post_5_8 = post_5 / (c << obs_8)
>>> post_5_8.plot()
```



The expected number of fish is now 89

Where we are, so far

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Effectuses

Basic results in effectus theory

Tool support for effectus probability

Conclusions

Main points

- Effectus is a basic notion of the “Nijmegen school”
  - weak axioms, but suprisingly rich (logical) structure
- Different primitives:
  - Oxford: tensors $\otimes$ and interaction, after Schrödinger
  - Nijmegen: coproducts $+$ and logic, after von Neumann

  There is “stronger entanglement of research”
- Basics of effectus theory is now well-developed:
  - state-and-effect triangles
  - commutative (probabilistic) and Boolean subcases
  - quotient and comprehension chains
  - conditioning (update,revision) of states with predicates
  - square root axiom, with pure maps and daggers
- EfProb tool support for discrete/continuous/quantum channel-based probability calculations
- Is effectus theory the ‘new topos theory’? Far too early to say!